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Thermal entanglement in two qutrits system

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Abstract. The thermal entanglement in a two-qutrit system with two spins coupled by exchange interaction is investigated in terms of the measure of entanglement called "negativity". It is found that the thermal entanglement is present and evolvements symmetrically between both ferromagnetic and anti-ferromagnetic exchange couplings with the temperature. Moreover the critical temperature at which the negativity vanishes increases with the exchange coupling constant J. From the temperature and magnetic field dependences we demonstrate that the temperature and the magnetic field can affect the feature of the thermal entanglement significantly.

PACS. 03.65.Ud Entanglement and quantum nonlocality (e.g. EPR paradox, Bell's inequalities, GHZ states, etc.) – 75.10.Jm Quantized spin models – 05.50.+q Lattice theory and statistics (Ising, Potts, etc.) – 03.67.Lx Quantum computation

Entanglement as a key concept in quantum information processing (QIP) [1–3] has attracted a lot of attention both experimentally and theoretically in recent years [4]. Since the entanglement is fragile, the problem of how to create stable entanglement remains a main focus of recent studies in the field of quantum information processing. The thermal entanglement, which differs from the other kinds of entanglements by its advantages of stability, requires neither measurement nor controlled switching of interactions in the preparing process, and hence becomes an important quantity of systems for the purpose of quantum computing.

The thermal entanglement has been extensively studied for various systems including isotropic [5–8] and anisotropic [9] Heisenberg chains, Ising model in an arbitrarily directed magnetic field [10], and cavity-QED [11] since the seminal works by Arnesen et al. [6] and Nielsen [12]. Based on the method developed in the context of quantum information, the relaxation of a quantum system towards the thermal equilibrium is investigated [13] and provides us an alternative mechanism to model the spin systems of the spin-$\frac{1}{2}$ case for the approaching of the thermal entangled states [5–9,14–17]. In this paper, we study the thermal entanglement, however, for two spin-1 particles system with two spins coupled by the exchange interaction in an external magnetic field. We derive the thermal entanglement as a function of the exchange constant, the temperature and the external magnetic field as well. The development of laser cooling and trapping provides us more ways to control the atoms in traps. Indeed, we can manipulate the atom-atom coupling constants and the atom number in each lattice well with a very well accuracy, our system consists of two wells in the optical lattice with one spin-1 atom in each well [18]. When the nonlinear couplings are ignored, the system in the absence of the external magnetic field can be described by the following Hamiltonian

$$H = J(S_1^x S_2^x + S_1^y S_2^y) \tag{1}$$

in which the neglected exchange coupling term along the z-axes is assumed to be much smaller than the coupling in the x–y-plane. Where S^α $(\alpha = x, y)$ are the spin operator, J is the strength of Heisenberg interaction and total spin for each site $S_i = 1$ $(i = 1, 2)$. With the help of raising and lowering operators $S_n^\pm = S_n^x \pm iS_n^y$, the Hamiltonian H is rewritten as

$$H = \frac{J}{2}(S_1^+ S_2^- + S_1^- S_2^+). \tag{2}$$

To evaluate the thermal entanglement we first of all find the eigenvalues and the corresponding eigenstates of the

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Hamiltonian (Eq. (2)) which are seen to be

$$\begin{aligned} H\left|1,1\right\rangle &= 0,\\ H\left|-1,-1\right\rangle &= 0,\\ H\left|\Psi_1\right\rangle &= 0,\\ H\left|\Psi_2^{\pm}\right\rangle &= \pm\frac{J}{2}\left|\Psi_2^{\pm}\right\rangle,\\ H\left|\Psi_3^{\pm}\right\rangle &= \pm\frac{J}{2}\left|\Psi_3^{\pm}\right\rangle,\\ H\left|\Psi_4^{\pm}\right\rangle &= \pm\frac{J}{\sqrt{2}}\left|\Psi_4^{\pm}\right\rangle. \end{aligned} \quad (3)$$

where

$$\begin{aligned} \left|\Psi_1\right\rangle &= \frac{1}{\sqrt{2}}(\left|1,-1\right\rangle - \left|-1,1\right\rangle),\\ \left|\Psi_2^{\pm}\right\rangle &= \frac{1}{\sqrt{2}}(\left|0,-1\right\rangle \pm \left|-1,0\right\rangle),\\ \left|\Psi_3^{\pm}\right\rangle &= \frac{1}{\sqrt{2}}(\left|1,0\right\rangle \pm \left|0,1\right\rangle),\\ \left|\Psi_4^{\pm}\right\rangle &= \frac{1}{\sqrt{2}}\left|-1,1\right\rangle \pm \left|0,0\right\rangle + \frac{1}{\sqrt{2}}\left|1,-1\right\rangle. \end{aligned} \quad (4)$$

Here $\left|i,j\right\rangle$ ($i=-1,0,1$ and $j=-1,0,1$) are the eigenstates of $S_1^z S_2^z$. The density operator at thermal equilibrium $\rho(T)=\exp(-\beta H)/Z$, where $Z=\mathrm{Tr}[\exp(-\beta H)]$ is the partition function and $\beta=1/k_BT$ (k_B is Boltzmann's constant being set to be unit $k_B=1$ hereafter for the sake of simplicity and T is the temperature), can be expressed in terms of the eigenstates and the corresponding eigenvalues as

$$\begin{aligned} \rho(T) = \frac{1}{Z}\Big\{ &\left|\Psi_1\right\rangle\left\langle\Psi_1\right| + \left|-1,-1\right\rangle\left\langle -1,-1\right| + \left|1,1\right\rangle\left\langle 1,1\right| \\ &+ \exp[m]\left|\Psi_2^+\right\rangle\left\langle\Psi_2^+\right| + \exp[-m]\left|\Psi_2^-\right\rangle\left\langle\Psi_2^-\right| \\ &+ \exp[m]\left|\Psi_3^+\right\rangle\left\langle\Psi_3^+\right| + \exp[-m]\left|\Psi_3^-\right\rangle\left\langle\Psi_3^-\right| \\ &+ \exp[\sqrt{2}m]\left|\Psi_4^+\right\rangle\left\langle\Psi_4^+\right| + \exp[-\sqrt{2}m]\left|\Psi_4^-\right\rangle\left\langle\Psi_4^-\right| \Big\}, \end{aligned} \quad (5)$$

with the partition function seeing to be $Z = 3 + 4\cos h[m] + 4\cos h[\sqrt{2}m]$ and $m = J/2T$.

Here will give the entanglement in terms of the measure of entanglement called "negativity" [19] which can be computed effectively for any mixed state of an arbitrary bipartite system, the negativity vanishes (i.e. negativity is equal to zero) for unentangled states. For our purpose to evaluate the negativity in what following we need to have a partially transposed density matrix ρ^{T_A} of original density matrix ρ with respect to the eigenbase of any one spin particle (say particle A) of our two-spin system which is found in the basis $\{\left|i,j\right\rangle, i=-1,0,1 \text{ and } j=-1,0,1\}$ as

$$\rho^{T_A} = \frac{1}{Z} \times \begin{pmatrix} 1 & 0 & 0 & 0 & M & 0 & 0 & 0 & Q-\frac{1}{2} \\ 0 & N & 0 & 0 & 0 & \sqrt{2}P & 0 & 0 & 0 \\ 0 & 0 & Q+\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N & 0 & 0 & 0 & \sqrt{2}P & 0 \\ M & 0 & 0 & 0 & 2Q & 0 & 0 & 0 & M \\ 0 & \sqrt{2}P & 0 & 0 & 0 & N & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Q+\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}P & 0 & 0 & 0 & N & 0 \\ Q-\frac{1}{2} & 0 & 0 & 0 & M & 0 & 0 & 0 & 1 \end{pmatrix} \quad (6)$$

where $M = \sinh[J/2T]$, $N = \cosh[J/2T]$, $P = \sinh[J/\sqrt{2}T]$, $Q = \cosh[J/\sqrt{2}T]$. The negativity as the entanglement measure [20] is defined by

$$N(\rho) = \frac{\left|\left|\rho^{T_A}\right|\right| - 1}{2} \quad (7)$$

where $\left|\left|\rho^{T_A}\right|\right| = \sqrt{\mathrm{Tr}[\rho^{T_A}]^{+}\rho^{T_A}}$ denotes the trace norm [21] of the density matrix ρ^{T_A} which is equal to the sum of the absolute eigenvalues of ρ^{T_A}. Although the negativity lacks a direct physical interpretation, it bounds two relevant quantities in quantum information processing — the channel capacity and the distillable entanglement. As the negativity is a computable measure of entanglement for biparticle system with any dimension, we here adopt it to measure the thermal entanglement.

Summing over the absolute eigenvalues of the density matrix ρ^{T_A} we have

$$\begin{aligned} N(\rho) = \frac{1}{2\left|Z\right|}\Bigg(&\frac{2\left|Z\right|}{\left|p\right|} + \frac{\left|Z\right|}{\left|q\right|} + \frac{\left|d_-\right| + \left|d_+\right|}{4} \\ &+ 2\left|\cosh[m] + \sqrt{2}\sinh[\sqrt{2}m]\right| \\ &+ 2\left|\cosh[m] - \sqrt{2}\sinh[\sqrt{2}m]\right| - \left|Z\right| \Bigg) \end{aligned} \quad (8)$$

where

$$\begin{aligned} p &= 4 + \frac{2 + 8\cosh[m]}{1 + 2\cosh[\sqrt{2}m]},\\ q &= -4 + \frac{18 + 8\cosh[m]}{3 - 2\cosh[\sqrt{2}m]},\\ d_{\pm} &= 1 \pm 6\cosh[\sqrt{2}m] \\ &\quad - \sqrt{(1 - 2\cosh[\sqrt{2}m])^2 + 32\sin^2[m]}. \end{aligned}$$

Figure 1 shows the plot of the negativity as a function of the exchange constant J for different temperature T. The state $\left|\Psi_4^-\right\rangle$ is seen to be the ground state for $J > 0$ while the state $\left|\Psi_4^+\right\rangle$ is the ground state for $J < 0$. The entanglement approaches its maximal value at zero temperature $T \to 0$ and decreases with increase of temperature due

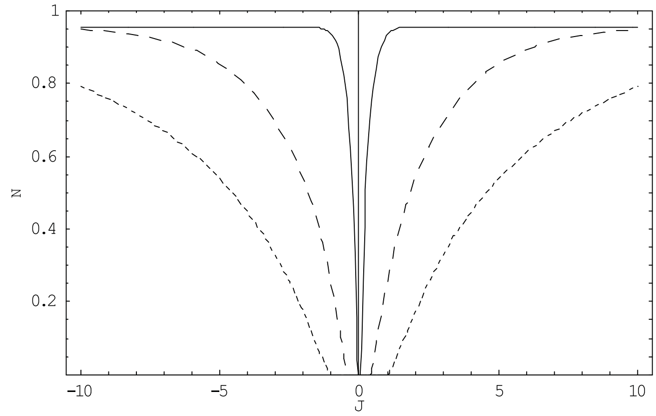

Fig. 1. The thermal entanglement vs. the exchange constant J for the different temperature T. $T = 0.05$ (solid line), $T = 0.4$ (dashed line), $T = 1$ (dotted line). J and T are plotted in units of the Boltzmann's constant k_B.

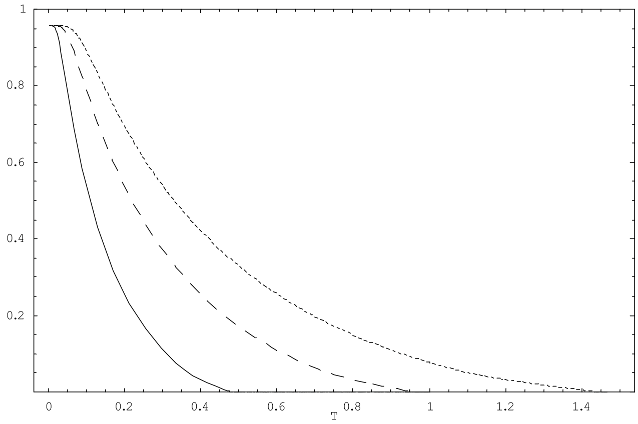

Fig. 2. The thermal entanglement vs. the temperature T for different exchange constant J. $J = 1$ (solid line), $J = 0.5$ (dashed line), $J = 1.5$ (dotted line).

to mixing of the excited states with the ground state. It is found that thermal entanglement is the same for both positive and negative exchange coupling J. That is to say, the entanglement is present in both antiferromagnetic and ferromagnetic chains. In contrary to this, for the case of two-qubit Heisenberg model, no thermal entanglement is present for the ferromagnetic case. We can see that there is a segment for J in which the negativity is zero, moreover the segment increases with T, these similar results have not been discussed for two-qubit case. From Figure 2, it can be seen that the critical temperature increases with J, that is to say, the entanglement at the fixed temperature can be increased by increasing the coupling constant J. In fact, the dependences of entanglement on the coupling constant are discussed for two-qubit XY models in the last studies [22], but that is only for the spin-$\frac{1}{2}$ case.

We now consider the model with an external magnetic field. With the help of raising and lowering operators the Hamiltonian H is written as

$$H = \frac{J}{2}(S_1^+ S_2^- + S_1^- S_2^+) + B(S_1^z + S_2^z). \tag{9}$$

The eigenvalues and eigenvectors are obtained as

$$\begin{aligned} H\left|1,1\right\rangle &= 2B\left|1,1\right\rangle, \\ H\left|-1,-1\right\rangle &= -2B\left|-1,-1\right\rangle, \\ H\left|\Psi_1\right\rangle &= 0, \\ H\left|\Psi_2^\pm\right\rangle &= \frac{1}{2}(-2B \pm J)\left|\Psi_2^\pm\right\rangle, \\ H\left|\Psi_3^\pm\right\rangle &= \frac{1}{2}(2B \pm J)\left|\Psi_3^\pm\right\rangle, \\ H\left|\Psi_4^\pm\right\rangle &= \pm\frac{J}{\sqrt{2}}\left|\Psi_4^\pm\right\rangle. \end{aligned} \tag{10}$$

The density operator is obtained as

$$\begin{aligned} \rho(T) = \frac{1}{Z}\Big\{ &\left|\Psi_1\right\rangle\left\langle\Psi_1\right| \\ &+ \exp[-2n]\left|-1,-1\right\rangle\left\langle-1,-1\right| + \exp[2n]\left|1,1\right\rangle\left\langle1,1\right| \\ &+ \exp[m-n]\left|\Psi_2^+\right\rangle\left\langle\Psi_2^+\right| + \exp[-n-m]\left|\Psi_2^-\right\rangle\left\langle\Psi_2^-\right| \\ &+ \exp[m+n]\left|\Psi_3^+\right\rangle\left\langle\Psi_3^+\right| + \exp[n-m]\left|\Psi_3^-\right\rangle\left\langle\Psi_3^-\right| \\ &+ \exp[\sqrt{2}m]\left|\Psi_4^+\right\rangle\left\langle\Psi_4^+\right| + \exp[-\sqrt{2}m]]\left|\Psi_4^-\right\rangle\left\langle\Psi_4^-\right| \Big\}, \end{aligned} \tag{11}$$

with the partition function $Z = 1 + 2\cos h[2n] + 4\cos h[n]\cos h[m] + 4\cos h[\sqrt{2}m]$, $n = B/T$.

see equation (12) below.

We perform the numerical diagonalization of the density matrix and the numerical results of the entanglement measure are presented in figures from 3 to 4. Figure 3 shows the plot of the negativity as a function of temperature T for various fixed values of magnetic field B when $J = 1$. For $B = 0$ and $J > 0$, the state $\left|\Psi_4^-\right\rangle$ is the ground state with eigenvalues $-J/\sqrt{2}$. In this case, the maximal entanglement is approached at $T = 0$. As T increases, the negativity decreases due to the mixing of the excited states

$$\rho^{T_A} = \frac{1}{Z}\begin{pmatrix} \exp[-2n] & 0 & 0 & 0 & \exp[-n]M & 0 & 0 & 0 & Q-\frac{1}{2} \\ 0 & \exp[-n]N & 0 & 0 & 0 & \sqrt{2}P & 0 & 0 & 0 \\ 0 & 0 & Q+\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \exp[-n]N & 0 & 0 & 0 & \sqrt{2}P & 0 \\ \exp[-n]M & 0 & 0 & 0 & 2Q & 0 & 0 & 0 & \exp[n]M \\ 0 & \sqrt{2}P & 0 & 0 & 0 & \exp[n]N & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Q+\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}P & 0 & 0 & 0 & \exp[n]N & 0 \\ Q-\frac{1}{2} & 0 & 0 & 0 & \exp[n]M & 0 & 0 & 0 & \exp[2n] \end{pmatrix} \tag{12}$$

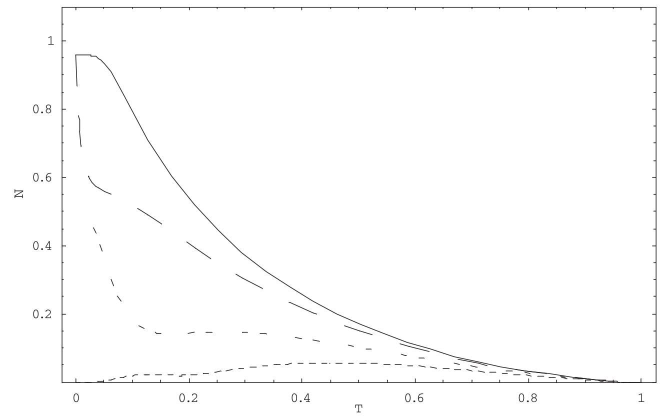

Fig. 3. The thermal entanglement vs. the temperature T for $J = 1$ and different magnetic field: $B = 0$ (solid line), $B = 0.2$ (long dashed line), $B = 0.4$ (dashed-dashed line), $B = 0.6$ (dotted line).

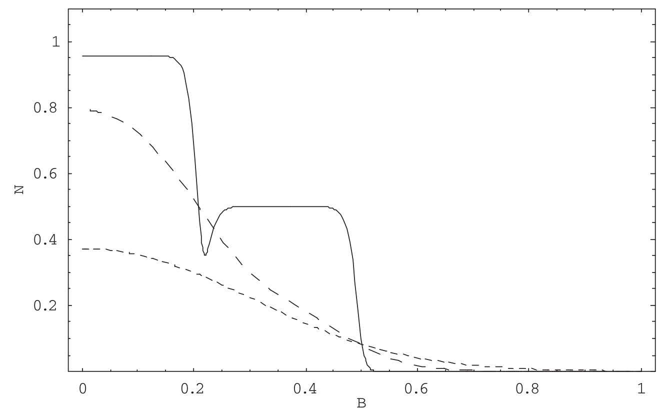

Fig. 4. The thermal entanglement vs. the magnetic field B for $J = 1$ and different temperature: $T = 0.01$ (solid line), $T = 0.1$ (dashed line), $T = 0.3$ (dotted line).

with the ground state. For a higher value of the magnetic field B (say $B = 0.6$), the state $|-1,-1\rangle$ becomes the ground state and there is no entanglement at $T = 0$. However we may increase the entanglement by increasing of temperature T in order to bring the entangled eigenstates such as $|\Psi_1\rangle$, $|\Psi_2^{\pm}\rangle$, $|\Psi_3^{\pm}\rangle$, $|\Psi_4^{\pm}\rangle$, into mixing with the ground state. We also found that the critical temperature is almost the same for different external field. These results are consistent with those for two spin-$\frac{1}{2}$ particle system. From Figures 3 and 4, for a fixed coupling constant (say $J = 1$), the entanglement with external magnetic field trails off to zero at a faster rate than that without external magnetic field, moreover, the maximum entanglement that the system can arrived becomes smaller as the magnetic field increases.

In Figure 4, the plot of the negativity as a function of the magnetic field B for different temperature T is presented. For $B = 0$, we can see that the lower temperature corresponds to the higher negativity. The negativity tardily decreases as the magnetic field increases. The change in negativity as T increases from absolute zero is due to population of excited levels. When the temperature is close to absolute zero, we can see that the negativity has a sharp change and becomes a nonanalytic function of B.

To conclude, we have studied the thermal entanglement in a two-qutrit system and the dependence of the thermal entanglement on the exchange constant, the external magnetic field and temperature as well. We have found that thermal entanglement is present for both ferromagnetic and antiferromagnetic coupling and moreover the negativity are the same for both positive and negative couplings. That is to say, the entanglement is present in both antiferromagnetic and ferromagnetic chains. In contrary to this, for the case of two-qubit Heisenberg model, no thermal entanglement is present for the ferromagnetic case. The entanglement at the fixed temperature can be increased by increasing the coupling constant J. The critical temperature is dependent of the exchange constant J and increases with the absolute value of $|J|$, moreover, the critical temperature is almost the same for different external field. These results are consistent with the previous study for spin-$\frac{1}{2}$ case. The entanglement with external magnetic field trails off to zero at a faster rate than that without external magnetic field. The change in negativity as T increases from absolute zero is due to population of excited levels. The results show that the temperature and the magnetic field really affect the feature of the thermal entanglement in the system considered.

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